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On Extension of Weibull Distribution with Bayesian Analysis using S-Plus Software

Bilal Ahmad Bhat
Associate Professor (Statistics), Division of Social Science, Faculty of Fisheries, SKUAST-Kashmir,Rangil Ganderbal, J&K, INDIA

Corresponding Author: bhat_bilal@rediffmail.com

ABSTRACT

We introduce a new lifetime distribution which is an extension of Weibull distribution and yields a number of well-known distributions as a special case. The Bayesian analysis using different priors is discussed using different priors with the help of S-Plus software as nowadays great attention has been given to Bayesian approach.

Keywords-- Weibull distribution, Bayesian method, Prior, Probability density function, Statistical model

I. INTRODUCTION

One of the most remarkable application of probability theory is to depict the real life phenomenon through probability model or distribution. Statistical models describe a phenomenon in the form of mathematical equations. In the literature (Hogg & Crag (1970), Johnson & Kotz (1970), Lawless (1982) etc.) we come across different types of models e.g., Linear models, Non-linear models, Generalized linear models, Generalized addition models, Analysis of variance models, Stochastic models, Empirical models, Simulation models, Diagnostic models, Operation research models, Catalytic models, Deterministic models, etc. Out of large number of methods and tools developed so far for analyzing data (on the life sciences etc.), the statistical models are the latest innovations.

Weibull distribution was originally derived in 1928 by Professor R.A.Fisher and L.H.C. Tippet and became known to researchers as the Fisher-Tippet Type III distribution. In 1939, Waloddi Weibull developed this distribution for analysis of breaking-strength data and he also published several other papers on the wide applicability of the distribution. The use of Weibull distribution in reliability and quality control is well known. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is some times used as a tolerance distribution in the analysis of quantal response data. The main justification for consideration of the Weibull distribution is that it has been shown experimentally to provide a good fit for many different types of characteristics since Waloddi Weibull derived it in an analysis of breaking strengths (Sinha (1986)). It is perhaps the most popular parametric family of distributions, being used as a failure model

In this paper, we propose a model which is an extension of the model proposed by Bilal (2011) and includes weibull distribution as a special case. Finally, we discuss Bayesian method of estimation in case of Weibull distribution. It is interesting to note that researchers are trying to find out which estimation method is preferable for the parameter estimation of a particular probability distribution in order to get reliable estimates of the parameters.

The most popular and easiest methods of utilizing data from a random sample to estimate the parameters of the Weibull distribution have been the log-log graphical plot, the Weibull graphical plot, least square estimates, moment estimates, maximum likelihood estimates and the several more complicated methods proposed by statisticians in technical journals.

II. BAYESIAN APPROACH

Bayesian Statistics is an approach to statistics which formally seeks use of prior information with the data, and Bayes Theorem provides the formal basis for making use of both sources of information in a formal manner. Bayes Theorem states that

Posterior ∝ likelihood x prior

The prior is the probability of the parameter and represents what was thought before seeing the data. The likelihood is the probability of the data given the parameter and represents the data now available.

The posterior represents what is thought given both prior information and the data just seen. It relates the conditional density of a parameter (posterior probability) with its unconditional density (prior, since depends on information present before the experiment).

or equivalently Bayes' Theorem for a given parameter θ is given by

$$p(\mid \theta \text{ data}) = p(\text{data } \theta \mid)\, p(\theta) / p(\text{data})$$

1/p (data) is basically a normalizing constant

In spite of problems, Bayesian methods are very popular with statisticians, even those who don't accept subjective Bayesian principles. Empirical Bayesian statisticians use conjugate priors (priors that lead to simple posteriors), uninformative priors (priors like the uniform distribution), and improper priors (priors like the uniform distribution on an infinite interval), to arrive at estimates like those of classical sampling theory.

III. PROPOSED MODEL

Let g(x) be a continuous monotonic increasing function in (k, ∞) such that g(∞)= ∞ and g(k)=0 where k is any positive real number, then the function

$$f(x; g, \alpha, \theta, \beta, \lambda) = \left(\alpha\theta\left(g(x)^{\theta-1}\right) + \beta\lambda g(x) e^{\lambda g(x)}\right) g'(x) \times \exp\left\{-\alpha\left(g(x)^{\theta}\right) - \beta e^{\lambda g(x)}\right\} \qquad (1.1)$$

where α, θ, β and λ are non-negative, with θ being shape parameter and α, and β being scale parameters and λ acceleration parameter. The function (1.1) can exhibit different behaviour depending on g(x) and the values of the parameters chosen.

In fact, a number of new of new and some well-known distributions follow the model(1.1) for a suitable choice of the function g(x) and the parameters of the proposed model. Here we mention Weibull model

Remark 1.1. Taking g(x)=x so that g´(x)=1 and β=0 in (1.1), we have

IV. DERIVATION OF WEIBULL DISTRIBUTION

$$f(x/\alpha, \theta) = \alpha\theta x^{\theta-1} \exp\left(-\alpha x^{\theta}\right);\ x,\ \alpha, \theta > 0$$
$$= 0 \text{ otherwise}$$

Which is the pdf of well-known two parameter Weibull distribution with parameters α >0 and ɵ>0.

Similarly as above, many more distributions can be shown as a particular case of the proposed statistical model (1.1) for a suitable choice of the generating function g(x) and the parameters.

V. METHODS OF ESTIMATION OF WEIBULL MODEL

The probability density function of Weibull model is given by

$$f(x/\alpha, \theta) = \frac{\alpha}{\theta} x^{\alpha-1} \exp\left(-\frac{x^{\alpha}}{\theta}\right);\ x,\ \alpha, \theta > 0$$
$$= 0 \text{ otherwise} \qquad (1.1.1)$$

If $x_1, x_2, \ldots, x_n$ be iid observations from weibull model, then the likelihood function is given by

$$L(x,\alpha,\theta) = \prod f(x,\alpha,\theta)$$

\ $$= \frac{\alpha^n}{\theta^n} \prod x_i^{\alpha-1} \exp\left(-\frac{\alpha^n}{\theta^{\alpha n}}\right)$$

The maximum likelihood estimator is given by

$$\theta^* = \frac{\sum x_i}{\sqrt[\alpha]{\alpha n}} \quad (1.1.2)$$

Suppose shape parameters α be known and let

$$g(\theta) \propto \frac{\exp(-a/\theta)}{\theta^c}, a, c > 1, \theta > 0 \quad (1.1.3)$$

be the prior distribution of θ. Then its posterior is

$$\prod(\theta/x) = K \frac{\exp\left(-\frac{\sum x_i^\alpha + a}{\theta}\right)}{\theta^{n+c}} \quad (1.1.4)$$

where

$$K^{-1} = \int_0^\infty \frac{\exp\left(-\frac{\sum x_i^\alpha + a}{\theta}\right)}{\theta^{n+c}} d\theta = \Gamma \frac{(n+c-1)}{\left(\sum x_i^\alpha + a\right)^{n+c-1}}$$

Bays Estimator of θ is given by

$$\theta^* = \frac{\sum x_i^\alpha + a}{n+c-1}$$

In case of Jeffrey's invaria nt prior, the prior distribution of (θ, α) is g(θ) h(α) where $g(\theta) \propto 1/\theta^c, c > 0$ and $h(\alpha) \propto 1/a, a < \alpha < a$. With these priors, the posterior distribution of (θ,α) is given by

$$\prod(\theta,\alpha/x) = \frac{K\alpha^n}{\theta^{n+c}} \lambda^{\alpha-1} \exp\left(\frac{\sum x_i^\alpha}{\theta}\right) \quad (1.1.5)$$

where

$$\lambda = \prod_{i=1}^n x_i \text{ and } K^{-1} = \int_0^\alpha \int_0^\infty \prod(\theta,\alpha/x) d\theta\, d\alpha = \Gamma(n+c-1) \int_0^\alpha \frac{\alpha^n \lambda^{\alpha-1}}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n+c-1}} d\alpha$$

The Marginal posterior of α is given by

$$\prod(\alpha / x) = \frac{\dfrac{\alpha^{n}\lambda^{\alpha-1}}{\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{n+c-1}}}{\int_0^{\alpha} \dfrac{\alpha^{n}\lambda^{\alpha-1}}{\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{n+c-1}} d\alpha} \qquad (1.1.6)$$

Under square error loss function $l(\hat{\theta}-\theta)=c(\hat{\theta}-\theta)$, the risk function is given by

$$R(\hat{\theta}-\theta) = c\hat{\theta}^2 - \frac{2c\hat{\theta}\sum_{i=1}^{n} x_i}{\left(\frac{n-1}{c}\right)} + \frac{c\left(\sum_{i=1}^{n} x_i\right)^2}{\left(\frac{n-1}{c}\right)\left(\frac{n-2}{c}\right)} \qquad (1.1.7)$$

and the Bays Estimator as $\hat{\theta}_2 = \dfrac{\sum_{i=1}^{n} x_i}{\left(\frac{n-1}{c}\right)}$

Now under the loss function $l(\hat{\theta}-\theta) = \sqrt{\theta}\ (\hat{\theta}-\theta)$, we obtain the Bays Estimators as

$$\hat{\theta}_3 = \frac{\sum_{i=1}^{n} x_i}{\left(\dfrac{n-\frac{3}{2}}{c}\right)}$$

VI. SIMULATION STUDY

In the simulation study, we have chosen n=35, 75, 100 to represent small, moderate and large sample size, several values of the parameters α = 0.5, 1, 1.5, 2 with four values of constants. The number of replication used was R=1000. The simulation program was written by using S-Plus/R software. After the parameter was estimated, mean square error (MSE) and mean percentage error MPE were calculated. We have

$$MSE(\hat{\alpha}) = \frac{\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)}{R} \quad and \quad MPE(\hat{\theta}) = \frac{\sum_{i=1}^{1000}\frac{|\hat{\alpha}-\alpha|}{\alpha}}{R}$$

VIII. RESULTS

Table 1 : Estimators with respect to MSE

Size of Sample	α	θ	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
35	0.5	0.4 0.8	0.2001 0.2001	0.1349 0.1320	0.2120 0.2120

	1	0.4	0.2221	0.2089	0.2287
		0.8	0.2221	0.2088	0.2288
	1.5	0.4	0.2343	0.2322	0.2335
		0.8	0.2342	0.2324	0.2336
75	0.5	0.4	0.1248	0.1213	0.1289
		0.8	0.1249	0.1215	0.1286
	1	0.4	0.1231	0.1201	0.1223
		0.8	0.1236	0.1203	0.1228
	1.5	0.4	0.1266	0.1249	0.1304
		0.8	0.1281	0.1261	0.1329
100	0.5	0.4	0.0887	0.0886	0.0939
		0.8	0.0888	0.0886	0.0938
	1	0.4	0.0886	0.0858	0.0874
		0.8	0.0887	0.0858	0.0873
	1.5	0.4	0.0886	0.0854	0.0888
		0.8	0.0887	0.0868	0.0889

Table 2 : Estimators with respect to MPE

Size of Sample	α	θ	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
35	0.5	0.4	0.0201	0.0134	0.0213
		0.8	0.0203	0.0131	0.0212
	1	0.4	0.0223	0.0202	0.0223
		0.8	0.0224	0.0203	0.0224
	1.5	0.4	0.0232	0.0232	0.0239
		0.8	0.0233	0.0232	0.0241
75	0.5	0.4	0.0123	0.0121	0.0128
		0.8	0.0124	0.0122	0.0141
	1	0.4	0.0127	0.0123	0.0149
		0.8	0.0129	0.0122	0.0148
	1.5	0.4	0.0123	0.0123	0.0132
		0.8	0.0124	0.0123	0.0133
100	0.5	0.4	0.0083	0.0074	0.0094
		0.8	0.0082	0.0073	0.0093
	1	0.4	0.0063	0.0052	0.0073
		0.8	0.0064	0.0053	0.0072
	1.5	0.4	0.0072	0.0054	0.0064
		0.8	0.0073	0.0053	0.0063

VIII. CONCLUSION

The results obtained from the simulation study presented in Table 1 and Table 2 reveals that Bayes estimator with general Jeffery prior information, is the best estimator when compared to standard Bayes and Maximum likelihood estimator. The estimator with smallest MSE is considered as best and with largest MSE it is considered as worst. It is concluded that MSE and MPE of Bayes estimators decrease with an increase of sample size.

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