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A Twin Result on Positive Solutions for Boundary Value Problems

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ABSTRACT

In this chapter, we generate a twin result on positive solutions for boundary value problems.

***Keywords*--** Value, Boundary, Theorem

I. INTRODUCTION

We present the following assumptions and lemmas that are used for proving our main theorem.

For a constant $\delta \in \left(0, \frac{1}{2}\right)$,

let

$$\sigma = \min\{\xi w(t) : \delta \le t \le 1-\delta\},$$

$$l = \|w\|,$$

$$P = \max_{0\le t\le 1} \int_{\delta}^{1-\delta} G(t,s)h(s)ds,$$

and

$$K_1 = \{x \in K : x(t) \ge \sigma\|x\|, \delta \le t \le 1-\delta\}.$$

LEMMA: B

Suppose (A1) holds, for $v(t) \in C([0,1])$, $v(t) \ge 0$, then the problem

$$x^{(n)}(t) + v(t) = 0, \qquad \rightarrow (1.1)$$

with the boundary conditions (1.2) - (1.4) has the unique solution

$$x(t) = \int_0^1 G(t,s)v(s)ds, \; t \in [0,1]. \qquad \rightarrow (1.2)$$

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LEMMA: C

For $(t,s)\in[0,1]\times[0,1]$, we have

$$\left.\begin{aligned} g(t,s)&\le Lg(t,t),\\ g(t,s)&\le Lg(s,s),\end{aligned}\right\} \quad \rightarrow (1.3)$$

where $L\ge 1$ is given by

$$L=\max\left\{1,\frac{\beta_1}{\beta_1+\alpha_1},\frac{\beta_2}{\beta_2+\alpha_2}\right\}. \quad \rightarrow (1.4)$$

LEMMA: D

If $x(t)$ is a solution of boundary value problem (1.1) – (1.4), then we have

$$x(t)\ge \xi\|x\|w(t)>0,\ t\in(0,1) \quad \rightarrow (1.5)$$

where

$$w(t)=\int_0^1 G(t,s)h(s)ds \quad \text{and}$$

$$\xi=\frac{\rho}{L^2\|h\|(\beta_1+\alpha_1)(\beta_2+\alpha_2)}>0.$$

PROOF:

Obviously, $w(t)$ is the unique solution of Equ. (4.1) with boundary conditions (1.2) – (1.4) for $v(t)\equiv h(t)$.

Then from lemma (B) and Equ. (1.1.1) and Equ. (1.3),

we have

$$x^{(n-2)}(t)=\int_0^1 g(t,s)v(s)ds$$

$$=\begin{cases}\displaystyle\int_0^1 \frac{1}{\rho}(\alpha_1 t+\beta_1)[\alpha_2(1-s)+\beta_2]v(s)ds, t\le s\\ \displaystyle\int_0^1 \frac{1}{\rho}(\alpha_1 s+\beta_1)[\alpha_2(1-t)+\beta_2]v(s)ds, s\le t\end{cases}$$

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$$\geq \begin{cases} \dfrac{\alpha_1 t+\beta_1}{\alpha_1+\beta_1}\displaystyle\int_0^1 \frac{1}{\rho}(\alpha_1 s+\beta_1)\left[\alpha_2(1-s)+\beta_2\right]v(s)ds, & t\leq s \\ \dfrac{\alpha_2(1-t)+\beta_2}{\alpha_2+\beta_2}\displaystyle\int_0^1 \frac{1}{\rho}(\alpha_1 s+\beta_1)\left[\alpha_2(1-s)+\beta_2\right]v(s)ds, & s\leq t \end{cases}$$

$$\geq \frac{(\alpha_1 t+\beta_1)\left[\alpha_2(1-t)+\beta_2\right]}{(\beta_1+\alpha_1)(\beta_2+\alpha_2)}\int_0^1 g(s,s)v(s)ds$$

$$\geq \frac{\|x\|}{L(\beta_1+\alpha_1)(\beta_2+\alpha_2)}(\alpha_1 t+\beta_1)\left[\alpha_2(1-t)+\beta_2\right]$$

$$=\frac{\rho.\|x\|}{L(\beta_1+\alpha_1)(\beta_2+\alpha_2)}g(t,t)$$

$$\geq \frac{\rho.\|x\|}{L^2(\beta_1+\alpha_1)(\beta_2+\alpha_2)}\int_0^1 g(t,s)ds$$

$$\geq \frac{\rho.\|x\|}{L^2\|h\|(\beta_1+\alpha_1)(\beta_2+\alpha_2)}\int_0^1 g(t,s)h(s)ds$$

$$=\xi\|x\|w^{(n-2)}(t).$$

Since $x(t)$ and $w(t)$ satisfy the boundary condition (1.2), then we have

$$x(t)=\int_0^t\int_0^{\tau_{n-3}}....\int_0^{\tau_1} x^{(n-2)}(s)dsd\tau_1.....d\tau_{n-3}$$

$$\geq \int_0^t\int_0^{\tau_{n-3}}....\int_0^{\tau_1}\left[\xi\|x\|w^{(n-2)}(s)\right]dsd\tau_1.....d\tau_{n-3}$$

$$=\xi\|x\|w(t)$$

Therefore $x(t)\geq\xi\|x\|w(t)>0$, for $0<t<1$.

Hence the proof of the lemma.

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THEOREM: 1.1

Assume that there exist some constants $d \geq 0$, $b_1 > a_1, a_2 \geq 0$, and $R > \sigma R > r + b_1 dl > r > M > 0$ such that

(i) $f(t,x) \geq -d$ for $0 \leq t \leq 1$, $M_1 w(t) \leq x \leq R$,

where $M_1 = \max\{M, r\xi\}$;

(ii) $$0 < \frac{M}{\min\limits_{0\leq t\leq 1} f(t, Mw(t))} = a_1 < b_1 = \frac{r}{N\left[d + \max\limits_{\substack{0\leq t\leq 1 \\ Mw(t)\leq x\leq r}} f(t,x)\right]};$$

(iii) $$0 < \frac{R}{P\left[d + \min\limits_{\substack{\delta\leq t\leq 1-\delta \\ \sigma R - b_1 dl\leq x\leq R}} f(t,x)\right]} = a_2.$$

Then boundary value problem (1.1) – (1.4) has at least twin positive solutions y_1 and y_2 satisfying

$$0 < Mw(t) \leq y_1(t), \|y_1\| < r, \text{ and } r \leq \|y_2\| < R, \ 0<t<1, \rightarrow (1.6)$$

provided that $\max\{a_1, a_2\} < \lambda < b_1$.

PROOF:

We define the auxiliary functions $F(t,x)$ and $F^*(t,x)$ as

$$F(t,x) = \begin{cases} f(t,x), & x \geq Mw(t), \\ f(t, Mw(t)), & x < Mw(t) \end{cases}$$

and $$F^*(t,x) = d + F(t, x - \lambda d w(t)). \qquad \rightarrow (1.7)$$

Then we have

$$\min_{0\leq t\leq 1} F^*(t, Mw(t)) = \min_{0\leq t\leq 1}\left[d + F(t, Mw(t) - \lambda d w(t))\right]$$

$$= \min_{0\leq t\leq 1}\left[d + F(t, Mw(t))\right] \rightarrow (1.8)$$

$$\geq \min_{0\leq t\leq 1} F(t, Mw(t))$$

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$$= \min_{0\le t\le 1} f(t, Mw(t))$$

Therefore

$$\min_{0\le t\le 1} F^*(t, Mw(t)) = \min_{0\le t\le 1} f(t, Mw(t)),$$

$$\max\{F^*(t,x):\ 0\le t\le 1,\ Mw(t)\le x\le r\}$$

$$=\max\{d+F(t,x-\lambda dw(t)):\ 0\le t\le 1,\ Mw(t)\le x\le r\}$$

$$=\max\{d+F(t,x):\ 0\le t\le 1,\ Mw(t)-\lambda dw(t)\le x\le r-\lambda dw(t)\}$$

$$\le\max\{d+F(t,x):\ 0\le t\le 1,\ Mw(t)\le x\le r\}$$

$$=\max\{d+f(t,x):\ 0\le t\le 1,\ Mw(t)\le x\le r\}$$

Therefore

$$\max\{F^*(t,x):\ 0\le t\le 1,\ Mw(t)\le x\le r\}$$

$$=\max\{d+f(t,x):\ 0\le t\le 1,\ Mw(t)\le x\le r\} \quad \to (1.9)$$

and

$$\min\{F^*(t,x):\ \delta\le t\le 1-\delta,\ \sigma R\le x\le R\}$$

$$=\min\{d+F(t,x-\lambda dw(t)):\ \delta\le t\le 1-\delta,\ \sigma R\le x\le R\}$$

$$=\min\{d+F(t,x):\ \delta\le t\le 1-\delta,\ \sigma R-\lambda dw(t)\le x\le R-\lambda dw(t)\}$$

$$\ge\min\{d+f(t,x):\ \delta\le t\le 1-\delta,\ \sigma R-b_1 dl\le x\le R\}$$

Therefore

$$\min\{F^*(t,x):\ \delta\le t\le 1-\delta,\ \sigma R\le x\le R\}$$

$$\ge\min\{d+f(t,x):\ \delta\le t\le 1-\delta,\ \sigma R-b_1 dl\le x\le R\} \to (1.10)$$

From condition (ii) and inequalities (4.8) and (4.9), we have

$$0<\frac{M}{\min_{0\le t\le 1} F^*(t, Mw(t))}\le a_1$$

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$$< b_1 \le \frac{r}{N \max\left\{F^*(t,x) : 0 \le t \le 1, Mw(t) \le x \le r\right\}}.$$

Then from Theorem (3.2.2) implies the equation

$$x^{(n)}(t) + \lambda h(t) F^*(t,x) = 0, \qquad \rightarrow (1.11)$$

with the boundary conditions (1.2) – (1.4) has a solution x_1, such that

$0 < Mw(t) \le x_1(t)$, $0 < t < 1$ and $\|x_1\| < r$ when $a_1 < \lambda < b_1$.

Let $F^{**}(t,x) = \max\left\{F^*(t,x), 0\right\}$ and consider the equation

$$x^{(n)}(t) + \lambda h(t) F^{**}(t,x) = 0, \qquad \rightarrow (1.12)$$

with the boundary conditions (1.2) – (1.4).

It is clear that a function $x = x(t)$ is a positive solution of Equ. (1.12) with (1.2) – (1.4) if x is a fixed point of the mapping $T : K_1 \to K_1$, where T is defined by

$$(Tx)(t) = \lambda \int_0^1 G(t,s) h(s) F^{**}(t, x(s)) ds, \quad x \in K_1.$$

Here T is a completely continuous operator.

Let

$$K_r^* = \left\{y \in K_1 : \|y\| < r\right\},$$

$$K_R^* = \left\{y \in K_1 : \|y\| = R\right\}.$$

Suppose $a_2 < \lambda < b_1$, for $x \in \partial K_r^*$,

set

$$J = \left\{t \in [0,1] : F^*(t, x(t)) \ge 0\right\}.$$

Then

$$(Tx)^{(n-2)}(t) = \lambda \int_0^1 g(t,s) h(s) F^{**}(s, x(s)) ds$$

$$= \lambda \int_J g(t,s) h(s) F^*(s, x(s)) ds$$

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$$< b_1 \int_J g(t,s)h(s)\max\left\{F^*(s,x): 0 \le s \le 1, x \le r\right\} ds$$

$$= b_1 \int_J g(t,s)h(s) \times \max\left\{F^*(s,x): 0 \le s \le 1, Mw(s) \le x \le r\right\} ds$$

$$\le b_1 \int_J g(t,s)h(s) \times \max\left\{d + F(s,x): 0 \le s \le 1, Mw(s) \le x \le r\right\} ds$$

$$\le b_1 \frac{r}{b_1 N} \int_J g(t,s)h(s) ds$$

$$= \frac{r}{N} \int_J g(t,s)h(s) ds \,.$$

Since $(Tx)(t)$ satisfy the boundary condition (1.2), then

$$(Ty)(t) \le \int_0^t \int_0^{\tau_{n-3}} \ldots \int_0^{\tau_1} \left[\frac{r}{N} \int_J g(v,s)h(s) ds \right] dv d\tau_1 \ldots\ldots d\tau_{n-3}$$

$$= \frac{r}{N} \int_J G(t,s)h(s) ds$$

$$\le r$$

$$= \|x\| .$$

Then we obtain that $\|Tx\| < \|x\|$, for $x \in \partial K_r^*$.

For $x \in \partial K_R^*$, using condition (iii) and inequality (1.10),

we have

$$(Tx)^{(n-2)}(t) = \lambda \int_0^1 g(t,s)h(s)F^{**}(s,x(s)) ds$$

$$> a_2 \int_\delta^{1-\delta} g(t,s)h(s)F^{**}(s,x(s)) ds$$

$$\ge a_2 \int_\delta^{1-\delta} g(t,s)h(s) \times \min\left\{F^{**}(s,x): \delta \le s \le 1-\delta, \sigma R \le x \le R\right\} ds$$

$$\ge a_2 \frac{R}{a_2 P} \int_\delta^{1-\delta} g(t,s)h(s) ds$$

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$$\geq \frac{R}{P}\int_{\delta}^{1-\delta} g(t,s)h(s)ds$$

Similarly, we have

$$(Ty)(t) \geq R = \|x\|.$$

Then

$$\|Tx\| \geq \|x\|, \text{ for } x \in \partial K_R^*.$$

It follows that Equ. (1.12) with the boundary conditions (1.2) – (1.4) has a solution x_2 such that

$$r < \|x_2\| < R.$$

From lemma (D), we obtain

$$x_2(t) \geq \xi \|x_2\| w(t)$$

$$> \xi r w(t),$$

which implies that $u_2(t)$ is also a solution of Equ. (1.11)

with (1.2) – (1.4).

Now we have shown that equation (1.11) with the boundary conditions (1.2) – (1.4) has two positive solutions x_1 and x_2 satisfying

$$0 < Mw(t) \leq x_1(t), \ \|x_1\| < r \leq \|x_2\| < R.$$

Finally we prove that $y(t) = x(t) - \lambda d w(t)$ is a positive solution of boundary value problem (1.1) – (1.4), when x is a positive solution of Equ. (1.11) with (1.2) – (1.4).

Let

$$x(t) = y(t) + \lambda d w(t).$$

Substituting the above value of $x(t)$, Equ. (4.11) becomes

$$y^{(n)}(t) + \lambda h(t)\left[F^*(t, y(t) + \lambda d w(t)) - d\right] = 0. \qquad \rightarrow (1.13)$$

From Equ.(4.7) we know that

$$F^*(t,x) = d + F(t, x - \lambda d w(t))$$

$$F^*(t,x) - d = F(t, x - \lambda d w(t)).$$

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Therefore

$$F^*(t,y(t)+\lambda dw(t))-d = F(t,y(t)+\lambda dw(t)-\lambda dw(t))$$

$$= F(t,y(t)). \qquad \rightarrow (1.14)$$

Substituting (1.14) in (1.13) we get

$$y^{(n)}(t)+\lambda h(t)F(t,y(t))=0. \qquad \rightarrow (1.15)$$

Since

$$\min_{0\le t\le 1} F(t,Mw(t))=\min_{0\le t\le 1} f(t,Mw(t)),$$

from the proof of inequality , we get

$$y(t)\ge Mw(t).$$

Then $y(t)$ is also a positive solution of boundary value problem (1.1) – (1.4).

Similarly, if we take

$$y_1(t)=x_1(t)-\lambda dw(t)$$

and

$$y_2(t)=x_2(t)-\lambda dw(t),$$

then we get $y_1(t)$ and $y_2(t)$ are also positive solutions of boundary value problem (1.1) –(1.4).

II. CONCLUSION

Therefore boundary value problem (1.1) – (1.4) has at least twin positive solutions y_1 and y_2 satisfying.

$$0<Mw(t)\le y_1(t),\ \|y_1\|<r, \text{ and } r\le\|y_2\|<R,\ 0<t<1,$$

provided that $\max\{a_1,a_2\}<\lambda<b_1$. Hence the proof of the theorem.

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