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Positive Solutions for Higher-Order Boundary Value Problems

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ABSTRACT

This papers is related to positive solutions for higher order boundary value problems.

Keywords— Boundary, Value, Function

I. INTRODUCTION

In this Paper, we generate the existence of positive solutions for boundary value problems

1.1 LEMMA

In this section, we present the following assumptions and lemma that are used for proving our theorems.

Let Banachspace $X = C^{n-2}[0,1]$,

cone $K = \{y \in X : y(t) \geq 0\}$.

Let $\|.\|$ denote the supremum norm on X, and for a constant c,

let

$$K_c = \{y \in K : \|y\| < c\},$$

$$\partial K_c = \{y \in K : \|y\| = c\}.$$

Let

$$N = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)h(s)ds.$$

We use also the following assumptions:

(A1) $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ and $\rho = \alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 > 0$;

(A2) $f \in C([0,1] \times [0,+\infty), R)$;

(A3) $h(t) \in C\left((0,1), R^+\right)$, $0 < \int_0^1 G(t,s)h(s)ds < +\infty$, where $R^+ = [0, +\infty)$, $G(t,s)$ is the Green's function of the problem $-x^{(n)} = 0$ with the boundary conditions (1.2) - (1.4).

It is clear that

$$g(t,s) = \frac{\partial^{n-2} G(t,s)}{\partial t^{n-2}}, \qquad \rightarrow (1.1)$$

is the Green's function of the boundary value problems

$$-x'' = 0, \quad 0 < t < 1, \qquad \rightarrow (1.2)$$

$$\left.\begin{aligned} \alpha_1 x(0) - \beta_1 x'(0) = 0, \\ \alpha_2 x(1) + \beta_2 x'(1) = 0, \end{aligned}\right\} \qquad \rightarrow (1.3)$$

given by,

$$g(t,s) = \begin{cases} \frac{1}{\rho}(\alpha_1 t + \beta_1)[\alpha_2(1-s) + \beta_2], & t \le s; \\ \frac{1}{\rho}(\alpha_1 s + \beta_1)[\alpha_2(1-t) + \beta_2], & s \le t. \end{cases} \qquad \rightarrow (1.4)$$

LEMMA: A

Suppose $T : X \rightarrow X$ is completely continuous. Define the operator $\theta : TX \rightarrow K$ by

$$(\theta y)(t) = \max\{y(t), w(t)\}, \text{ for } y \in TX,$$

where $w(t) \in C^{n-1}[0,1]$, $w(t) \ge 0$ is a given function. Then

$$\theta \text{o} T : X \rightarrow K$$

is also a completely continuous operator.

II. BVP

We suppose that (A1), (A2), (A3) hold. The following Theorems (2.1) and (2.2) give sufficient conditions which guarantee the existence of positive solutions for BVP (1.1) – (1.4).

THEOREM: 1.2.1

Assume there exist constants $r > M > 0$, such that

$$0 < \frac{M}{\min\limits_{0 \le t \le 1} f(t, Mw(t))} = a < b = \frac{r}{N \max\limits_{\substack{0 \le t \le 1 \\ Mw(t) \le x \le r}} f(t,x)}. \rightarrow (1.1)$$

Then BVP (1.1) – (1.4) has atleast one positive solution

$y(t)$ satisfying

$$0 < Mw(t) \le y(t), \quad 0 < t < 1 \quad and \quad \|y\| < r, \qquad \rightarrow (1.2)$$

provided that $\lambda \in [a, b)$.

PROOF:

We define the auxiliary function $F(t, x)$ as

$$F(t,x) = \begin{cases} f(t,x), & x \ge Mw(t), \\ f(t, Mw(t)), & x < Mw(t). \end{cases} \rightarrow (1.3)$$

Let the operator $T : K \rightarrow K$ be defined by

$$(Tx)(t) = \lambda \int_0^1 G(t,s)h(s)F(t, x(s))ds, \quad 0 \le t \le 1. \rightarrow (1.4)$$

Then T is on K a completely continuous operator.

Let the operator $\theta : X \rightarrow K$ be defined by

$$(\theta y)(t) = \max\{y(t), 0\}. \qquad \rightarrow (1.5)$$

From lemma (A),

$\theta o T : K \rightarrow K$ is also completely continuous.

For $x \in \partial K_r$, set

$$J = \{t \in [0,1] : F(t, x(t)) \ge 0\}.$$

Then we have,

$$(\theta o T)x(t) = \max\{(Tx)(t), 0\}$$

$$= \max\left\{\lambda \int_0^1 G(t,s)h(s)F(t,x(s))ds, 0\right\}$$

$$\leq \lambda \int_0^1 G(t,s)h(s)F(t,x(s))ds$$

$$\leq \lambda \int_J G(t,s)h(s)F(t,x(s))ds$$

$$< b \max_{\substack{0\leq t\leq 1 \\ 0\leq x\leq r}} F(t,x) \int_J G(t,s)h(s)ds$$

$$\leq Nb \max_{\substack{0\leq t\leq 1 \\ Mw(t)\leq x\leq r}} f(t,x)$$

$$\leq r.$$

Then for every $x \in \partial K_r$,

$$(\theta \text{o} T)x(t) \neq x,$$

it follows that

$$\deg_K \{I - \theta \text{o} T, K_r, 0\} = 1,$$

where $\deg_K$ stands for the degree in cone K.

Then $\theta \text{o} T$ has a fixed point $y \in K_r$.

To finish the proof based on the definition of F, it suffices to show that the fixed point $y \in K_r$ satisfying

$$(Ty)\ (t) \geq Mw(t),\quad 0 \leq t \leq 1, \qquad \rightarrow (1.6)$$

since $F = f$ in the region.

In order to show (3.2.6) is hold, we first show that

$$(Ty)^{(n-2)}(t) \geq Mw^{(n-2)}(t),\ 0 \leq t \leq 1. \quad \rightarrow (1.7)$$

Otherwise, let

$$u(t) = Mw^{(n-2)}(t) - (Ty)^{(n-2)}(t),\ 0 \leq t \leq 1,$$

then there exists $t_0 \in [0,1]$ such that

$$u(t_0) = \max_{0\leq t\leq 1}\{u(t)\} = A > 0. \qquad \rightarrow (1.8)$$

If $t_0 = 0$, then

$$u'(t) = (n-2)Mw^{(n-3)}(t) - (n-2)(Ty)^{(n-3)}(t)$$

$$u'(t_0)=(n-2)Mw^{(n-3)}(t_0)-(n-2)(Ty)^{(n-3)}(t_0)$$

Therefore $u'(0)\le 0$.

Since both $Mw(t)$ and $(Ty)(t)$ satisfy the boundary

condition (1.3), we have

$$\alpha_1 u(0)-\beta u'(0)$$

$$=\alpha_1\left[Mw^{(n-2)}(0)-(Ty)^{(n-2)}(0)\right]-\beta_1\left[Mw^{(n-1)}(0)-(Ty)^{(n-1)}(0)\right]$$

$$=\alpha_1 Mw^{(n-2)}(0)-\alpha_1(Ty)^{(n-2)}(0)-\beta_1 Mw^{(n-1)}(0)+\beta_1(Ty)^{(n-1)}(0)$$

$$=\left[\alpha_1 Mw^{(n-2)}(0)-\beta_1 Mw^{(n-1)}(0)\right]-\left[\alpha_1(Ty)^{(n-2)}(0)-\beta_1(Ty)^{(n-1)}(0)\right]$$

$$=0.$$

Therefore $\alpha_1 u(0)-\beta u'(0)=0$.

If $\beta_1=0$, from $\rho>0$, then $\alpha_1>0$, so $u(0)=0$, which contradicts to (1.2.8).

Then

$\alpha_1=0$, $\beta_1>0$ and $u'(0)=0$. →(1.9)

Now we claim that

$$u(t)>0,\ t\in[0,1]. \quad \rightarrow(1.10)$$

If the assertion is false, then there is $t_1\in(0,1]$ such that $u(t)>0, t\in[0,t_1)$, $u(t_1)=0$. →(1.2.11)

So for every $t\in(0,t_1]$, from (3.2.9), we have

$$u'(t)=u'(0)+\int_0^t u''(s)\,ds$$

$$=\int_0^t\left[Mw^{(n)}(s)-(Ty)^{(n)}(s)\right]ds$$

$$=-\int_0^t h(s)\left[M-\lambda F(t,y(s))\right]ds$$

$$\ge 0.$$

That is $u'(t) \geq 0$, $t \in [0, t_1)$, and from (1.2.11) we have the following contradiction

$$0 = u(t_1) \geq u(0) = L > 0.$$

Then Equ. (2.10) is hold.

If $t_0 = 1$, we can obtain (2.10) in a similar way.

Finally, if $t_0 \in (0,1)$ then $u'(t_0) = 0$. We are able to show that $u'(t) \geq 0$ respectively in $[0, t_0]$ and in $[t_0, 1]$ in the same way as the above argument. So (3.2.10) holds for all the possible cases.

Since both $Mw(t)$ and $(Ty)(t)$ satisfy the boundary condition (1.2)- (1.4), that is,

$$Mw^{(i)}(0) = (Ty)^{(i)}(0) = 0, \; i = 0,1,\ldots,n-3.$$

From (3.2.10), we have

$$Mw(t) - (Ty)(t) = \int_0^t \int_0^{\tau_{n-3}} \ldots \int_0^{\tau_1} u(s)\,ds\,d\tau_1 \ldots d\tau_{n-3}$$

$$> 0. \qquad \rightarrow (1.12)$$

Then we have the following contradiction

$$0 < Mw(t_0) - (Ty)(t_0)$$

$$= \int_0^1 G(t_0, s)h(s)M\,ds - \lambda \int_0^1 G(t_0, s)h(s)F(s, y(s))\,ds$$

$$= \int_0^1 G(t_0, s)h(s)\left[M - \lambda F(s, y(s))\right]ds$$

$$\leq \left[M - a \min_{0 \leq t \leq 1} f(t, Mw(t))\right] \int_0^1 G(t_0, s)h(s)\,ds$$

$$\leq 0.$$

Then $(\theta \circ T)y = Ty = y$ and y is a solution of boundary value problem (1.1)-(1.4).

Hence the proof of the theorem.

THEOREM: 3

Assume $f(t,0) \geq 0$, $h(t)f(t,0) \not\equiv 0$ and there exists $r > 0$, such that

$$b = \frac{r}{N \min\limits_{\substack{0 \le t \le 1 \\ 0 \le x \le r}} f(t,x)} > 0. \qquad \rightarrow (1.13)$$

Then when $\lambda < b$ boundary value problem(1.1)- (1.4) has atleast one positive solution $y(t)$ satisfying $0 < \|y\| < r$.

PROOF:

Let $$F(t,x) = \begin{cases} f(t,x), & x \ge 0, \\ f(t,0) - x, & x < 0. \end{cases} \qquad \rightarrow (1.14)$$

Similar to the proof of Theorem (1.2.1), we show $\theta \circ T$ has a fixed point $y \in K_r$, where K_r and T defined as in Theorem (1.2.1).

Now we claim that

$$(Ty)^{(n-2)}(t) \ge 0, \qquad 0 \le t \le 1. \qquad \rightarrow (1.15)$$

Otherwise, then there exists $t_0 \in [0,1]$ such that $(Ty)^{(n-2)}(t_0) = \min\limits_{0 \le t \le 1}\{(Ty)^{(n-2)}(t)\}$

$$= -B$$

$$< 0.$$

We prove that

$$(Ty)^{(n-2)}(t) < 0, \; 0 \le t \le 1. \qquad \rightarrow (1.16)$$

If $t_0 \in (0,1)$, then

$$(Ty)^{(n-1)}(t_0) = 0.$$

If (3.2.16) does not hold, then there is $t_1 \in [0, t_0) \cup (t_0, 1]$ satisfying

$$(Ty)^{(n-2)}(t_1) = 0, \quad \text{and}$$

$$(Ty)^{(n-2)}(t) < 0, \; t \in (t_1, t_0) \text{ or } t \in (t_0, t_1). \rightarrow (1.17)$$

Without loss of generality we suppose $t_1 \in [0, t_0)$. Then from boundary condition (1.2), for every $t \in (t_1, t_0)$, we have

$$(Ty)(t) = \int_0^t \int_0^{\tau_{n-3}} \cdots \int_0^{\tau_1} (Ty)^{n-2}(s)\, ds\, d\tau_1 \ldots d\tau_{n-3}$$

$$< 0. \qquad \rightarrow (1.18)$$

From (3.2.18), for every $t \in (t_1, t_0)$

$$(Ty)^{(n-1)}(t) = (Ty)^{(n-1)}(t_0) - \int_t^{t_0} (Ty)^{(n)}(s)ds$$

$$= \lambda \int_t^{t_0} h(s)F(t, y(s))ds$$

$$\geq 0,$$

Which implies the following contradiction

$$0 = (Ty)^{(n-2)}(t_1) = (Ty)^{(n-2)}(t_0) + \int_{t_0}^{t_1} (Ty)^{(n-1)}(s)ds$$

$$= -B - \int_{t_1}^{t_0} (Ty)^{(n-1)}(s)ds$$

$$\leq -B$$

$$< 0.$$

Then (1.2.15) is hold.

So for $t \in [0,1]$, we have

$$(Ty)(t) = \int_0^t \int_0^{\tau_{n-3}} .... \int_0^{\tau_1} (Ty)^{(n-2)}(s)dsd\tau_1.....d\tau_{n-3}$$

$$\geq 0.$$

If $t_0 = 0$ or $t_0 = 1$, with use of the boundary conditions we can show the above assertion in a similar way in Theorem (1.1).

Then

$$y = (\theta \circ T)y = Ty,$$

that is $y(t)$ is a non negative solution of boundary value problem (1.1) - (1.4) with $0 \leq \|y\| \leq r$.

III. CONCLUSION

Besides $h(t)f(t,0) \not\equiv 0$ implies $y(t) \not\equiv 0$ in $[0,1]$.

Therefore $0 \leq \|y\| \leq r$.

Hence the proof of the theorem.

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