Analytical Mechanics of Magnetic Particles Suspended in Magnetorheological Fluid

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ABSTRACT

In this paper, the behavior of MR particles has been systematically investigated within the scope of analytical mechanics. . A magnetorheological fluid belongs to a class of smart materials. In magnetorheological fluids, the motion of magnetic particles is controlled by the action of internal and external forces. This paper presents analytical mechanics for the interaction of system of particles in MR fluid. In this paper, basic principles of Analytical Mechanics are utilized for the construction of equations.

Keywords-- Magnetorheological Fluids, Smart Material, Hamiltonian Equations

I. INTRODUCTION

Magnetorheological fluid is the one of the important type of smart materials and when subjected to a magnetic force, these fluids greatly increase its apparent viscosity [1]. Among all the smart materials, the magnetorheological fluids are important group. Magnetorheological fluids are also referred as intelligent class of materials. Typically, magnetorheological fluids are consists of soft paramagnetic or ferromagnetic particles dispersed in a base fluid. The magnetic force makes the suspended particles adjust in chain along the magnetic field lines in a way to decrease the overall energy of the magnetic force as shown in Figure 2,

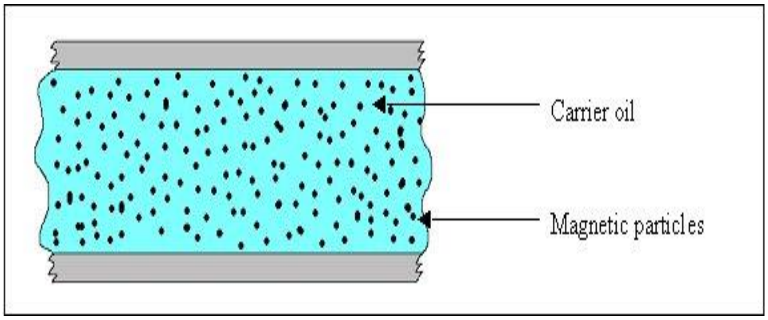

Figure 1: Iron particles present in Base fluid

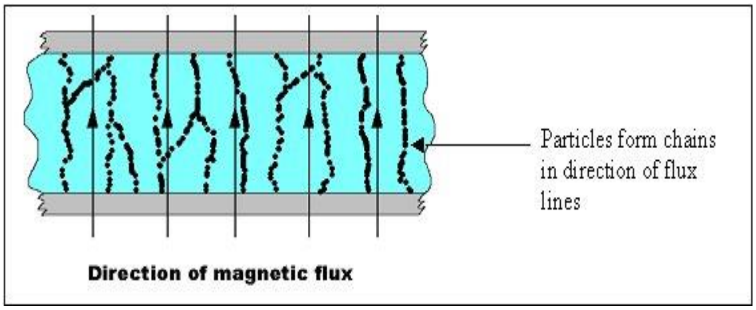

Figure 2: Behaviour of iron particles with Magnetic field

A Magnetorheological (MR) fluid is controlled by using external magnetic force and MR fluid changes from a liquid to a semi-solid state reversibly. Alternatively, in the absence of a magnetic force, MR fluids act as a

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Newtonian fluid. In conclusion, from fluid dynamics point of view the behaviour of MR fluids in the absence of magnetic force is considered as Newtonian fluid, while it illustrates distinct Bingham behaviour by the application of the magnetic force [2,3]. So that, MR fluid has been demonstrated in general as a Bingham fluid model, whose constitutive relation is represented by the following equation [4],

$$\tau = \tau_y(H) + n\dot{\gamma} \qquad \dots \quad (1)$$

Where, n = Dynamic Viscosity of fluid,
$\dot{\gamma}$ = Shear rate,
$\tau_y(H)$ = Dynamic yield stress of magnetorheological fluid,
τ = Shear stress.

Conventional magnetorheological fluids comprised of a base fluid, immersed with ferromagnetic micron-sized iron particles. By applying a magnetic field on MR fluids, the iron particle forms a chainlike structure in the direction of magnetic field lines. Therefore, the stronger the magnetic flux on the MR fluids, the stronger the particle chains, ultimately it increases yield strength of magnetorheological fluid. The magnetorheological fluid is consists of three main components: Base (Carrier) fluid, iron particles, and stabilizing additives [5].

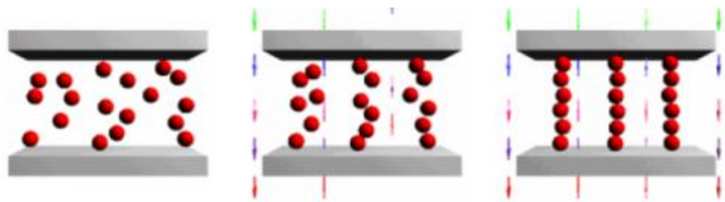

Figure 3: MR fluid Chain Formation

II. EQUATION OF MOTION FOR SYSTEM OF PARTICLES

Magnetorheological fluids are generally consist of soft paramagnetic or ferromagnetic particles dispersed in a base fluid and that exhibit dramatic changes in rheological properties, when subjected to external magnetic force. The magnetic force makes the suspended particles adjust in a chain along the magnetic field lines in a way to decrease the overall energy of the magnetic force [6, 7].

Suppose MR fluid contains n point magnetic particles of $masses$ $m_1, m_2, m_3, \dots, m_n$ having position vectors, $\bar{r}_i$, $i = 1,2,3, \dots, n$ with reference to the point O.

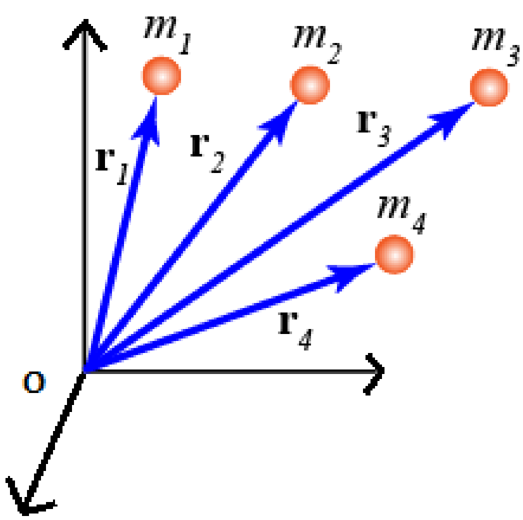

Figure 4: Positions of Particles in MR fluids

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The system of magnetic particles of the MR fluid undergoes three kinds of forces namely magnetic forces, external forces, and internal forces.

The overall force acting on the i^{th} particle of the system is given by the equation,

$$\bar{F}_i = \bar{F}_i^{(e)} + \bar{F}_{ji}^{(int)} + \bar{F}_i^{(M)}$$

Where $\bar{F}_i^{(e)}$ = Total External force , $\bar{F}_{ji}^{(int)}$ = Total Internal force and $\bar{F}_i^{(M)}$ = Total Magnetic force experienced by i^{th} particle.

Thus, according to Newton's second law of motion [7, 8],

$$\bar{F}_i^{(e)} + \sum_j \bar{F}_{ji}^{(int)} + \bar{F}_i^{(M)} = \frac{dp_i}{dt} \quad \dots (2)$$

By principle of superposition the overall interaction of i^{th} particle with remaining $(n-1)$ particles of the system is,

$$\sum_i \bar{F}_i^{(e)} + \sum_i \left(\sum_j \bar{F}_{ji}^{(int)} \right) + \sum_i \bar{F}_i^{(M)} = \sum_i \frac{dp_i}{dt}$$

This equation also can be written as,

$$\sum_i \bar{F}_i^{(e)} + \sum_{i,j,i\neq j} \bar{F}_{ji}^{(int)} + \sum_i \bar{F}_i^{(M)} = \sum_i \frac{dp_i}{dt} \quad \dots (3)$$

According to Newton's third law of motion,

$$\sum_{i,j,i\neq j} \bar{F}_{ji}^{(int)} = 0$$

Thus, equation (3) becomes,

$$\sum_i \bar{F}_i^{(e)} + 0 + \sum_i \bar{F}_i^{(M)} = \sum_i \frac{dp_i}{dt} \quad \dots (4)$$

$$\sum_i \bar{F}_i^{(e)} + \sum_i \bar{F}_i^{(M)} = \sum_i \frac{dp_i}{dt} \quad \dots (5)$$

Thus equation (5) represents the equation of motion for the system of particles in MR fluid.

The set of Lagrangian equations for the motion of a charged particle under electromagnetic force is [9, 10, 11],

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0, \quad k = 1,2,3,\dots,N \quad \dots (6)$$

Where R is Rayleigh's dissipation function and

$$L = T - V = T - q(\emptyset - v.A) \quad \dots (7)$$

III. HAMILTONIAN FUNCTION AND PRINCIPLE FOR MR FLUIDS

The Hamiltonian function is denoted by H and defined as,

$$H = \sum_k p_k \dot{q}_k - L \quad \dots (8)$$

Where p_k is the generalized momentum and derived by the relation [12],

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \quad \dots (9)$$

Also by the Lagrangian equation of motion, we can find,

$$\dot{p}_k = \frac{\partial L}{\partial q_k} \quad \dots (10)$$

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The Hamiltonian function H is the function of generalized momentum p_k , generalized coordinate q_k and time t. Hence, we can write,

$$H = H(p_k, q_k, t) = \sum_k p_k \dot{q}_k - L \quad \dots (11)$$

If particular generalised coordinate q_k does not appear in Lagrangian L explicitly. Then the value of

$$\frac{\partial L}{\partial q_k} = 0 \quad \dots (12)$$

From equation (12), equation (10) becomes,

$$\dot{p}_k = 0 \quad \dots (13)$$

By taking integration on both the sides, we get,

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{Constant} \quad \dots (14)$$

Thus, if particular generalised coordinate q_k does not appear in Lagrangian L explicitly, then the generalised momentum p_k is a constant of motion and such type of generalised coordinate q_k is called cyclic or ignorable coordinate.

Hamilton's principle for non-conservative and holonomic MR fluid states that "The motion of a dynamical MR fluid from time t_1 to time t_2 is such that the line integral has stationary values for the actual path followed by the MR fluid".

$$I = \int_{t_1}^{t_2} L\, dt \quad \text{with } L = T - V \quad \dots (15)$$

The term I is called the Hamiltonian principal function, T is the total kinetic energy and V is the total potential energy of the magnetorheological fluid.

The above Hamiltonian principle may be expressed as,

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0 \quad \dots (16)$$

where δ is the variation in the actual path followed by the MR fluid.

Now we derive Lagrange's equations of motion for conservative MR fluid by using Hamiltonian principle.

Let's consider conservative dynamical MR fluid whose configuration at any instant t is specified by the generalized coordinates $q_1, q_2, q_3, \dots, q_N$.

Then the Lagrangian for specified MR fluid is a function of generalized coordinates $q_1, q_2, q_3, \dots, q_N$, generalized velocities $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_N$ and time t.
Hence,

$$L = L(q_1, q_2, q_3, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N, t) \quad \dots (17)$$

If the Lagrangian does not depend on time t explicitly, then the variation δL can be stated as,

$$\delta L = \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \quad \dots (18)$$

Integrating both sides from t_1 *to* t_2 with respect to t, we get

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt \quad \dots (19)$$

We know Hamiltonian principle for conservative MR fluid [13],

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0$$

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This implies that,

$$\int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k \; dt + \int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \; dt = 0 \quad \dots (20)$$

Since we have,

$$\frac{d}{dt}(\delta q_k) = \delta \dot{q}_k \quad \dots (21)$$

Therefore, the above integral can be written as,

$$\int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k \; dt + \int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}(\delta q_k) \; dt = 0 \quad \dots (22)$$

Integrating by parts the second term on the left hand side of the above integral, we get

$$\int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k \; dt + \sum_{k=1}^{N} \left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right)_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \; dt = 0 \quad \dots (23)$$

Since δ variation there is no change in the coordinates at the end points $\Rightarrow (\delta q_k)_{t_1}^{t_2} = 0$.
Hence,

$$\left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right)_{t_1}^{t_2} = 0 \quad \forall k \Rightarrow \sum_{k=1}^{N} \left(\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right)_{t_1}^{t_2} = 0 \quad \dots (24)$$

Thus

$$\int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k \; dt - \int_{t_1}^{t_2} \sum_{k=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \; dt = 0 \quad \dots (25)$$

$$\sum_{k=1}^{N} \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] \delta q_k dt = 0 \quad \dots (26)$$

$$\int_{t_1}^{t_2} \sum_{k=1}^{N} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] \delta q_k dt = 0 \quad \dots (27)$$

If the constraints on magnetorheological fluids are holonomic then the generalized coordinates δq_k are independent of each other. Hence, the above integral vanishes if and only if the coefficients of each δq_k must vanish separately i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0, k = 1,2,3, \dots, N \quad \dots (28)$$

These are the Lagrangian equations of motion for conservative holonomic MR fluids and which is derived by Hamiltonian principle.

IV. HAMILTONIAN EQUATIONS FOR MR FLUIDS

The Lagrangian equations for motion of MR fluid are the set of second order ordinary differential equations. Here we derive Hamiltonian equation of motions for MR fluid in connection with generalized momentum by using Hamiltonian function and Hamiltonian principle. These equations of motion is more fundamental to the foundations of quantum mechanics and analytical mechanics.

We know the Hamiltonian function H is a function of the function of generalized momentum p_k, generalized coordinate q_k and time t. Hence, we can write,

$$H = H(p_1, p_2, p_3, \dots, p_N, q_1, q_2, q_3, \dots, q_N, t) \quad \dots (29)$$

$$or\ H = H(p_k, q_k, t), \qquad k = 1,2,3, \dots, N \quad \dots . (30)$$

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By definition of total derivative, we can write,

$$dH = \sum_{k=1}^{N} \frac{\partial H}{\partial p_k} dp_k + \sum_{k=1}^{N} \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \quad \dots (31)$$

By definition of Hamiltonian function,

$$H = \sum_k p_k \dot{q}_k - L \quad \dots .(32)$$

Again by definition of total derivative,

$$dH = \sum_k \dot{q}_k \, dp_k + \sum_k p_k \, d\dot{q}_k - dL$$

But from equation (17),

$$L = L(q_1, q_2, q_3, \dots, q_N, \dot{q}_1, \dot{q}_2, \dot{q}_3, \dots, \dot{q}_N t)$$

Hence,

$$dH = \sum_k \dot{q}_k \, dp_k + \sum_k p_k \, d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt \quad \dots (33)$$

But from equation (9),

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

Hence, equation (33) reduces to,

$$dH = \sum_k \dot{q}_k \, dp_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \quad \dots (34)$$

Now comparing the coefficients of dp_k , dq_k and dt in equations (31) and (34), we get,

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k}, \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \quad \dots (35)$$

By definition of generalized momentum and Lagrangian equation of motion we find,

$$\dot{p}_k = \frac{\partial L}{\partial q_k}$$

Equation (35) becomes,

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \dots (36)$$

Thus the equation (36) is called Hamilton's canonical equations of motion and which is the set of '$2N$' 1^{st} order partial differential equations and which explain the nature of magnetorheological fluid under electromagnetic force by taking

$$L = T - V = T - q(\emptyset - v.A)$$

Considering the MR fluid consists of n particles and N generalized coordinates then the Lagrangian function for the particle of mass m of the MR fluid is,

$$L = \frac{1}{2} \sum_{k=1}^{N} m{v^2}_k + q \sum_{k=1}^{N} v_k A_k - q\emptyset \quad \dots (37)$$

Suppose the MR fluid is placed in three-dimensional Cartesian coordinate system, then $k = x, y, z$.

Hence, above equation becomes,

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right) - q\emptyset \quad \dots (38)$$

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Here $\emptyset$ is a function of coordinate x, y, z only. In this system x, y, z are the generalized coordinates. Hence, the generalized momenta become,

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

$$\Rightarrow p_x = m\dot{x} + qA_x \,, p_y = m\dot{y} + qA_y \,, p_z = m\dot{z} + qA_z \quad \dots(39)$$

Solving these equations for $\dot{x}, \dot{y}, \dot{z}$ we get,

$$\dot{x} = \frac{1}{m}[p_x - qA_x], \dot{y} = \frac{1}{m}[p_y - qA_y], \dot{z} = \frac{1}{m}[p_z - qA_z] \quad \dots(40)$$

The Hamiltonian of the particle is given by,

$$H = \sum_k p_k \dot{q}_k - L$$

$$H = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) + q\emptyset \quad \dots(41)$$

From equation (40), equation (41) becomes,

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{q}{m}(p_xA_x + p_yA_y + p_zA_z) + \frac{1}{2m}q^2\,(A_x^2 + A_y^2 + A_z^2) + q\emptyset \quad \dots(42)$$

Then above Hamiltonian can be stated in vector form as,

$$H = \frac{1}{2m}(\bar{p} - q\bar{A})^2 + q\emptyset \quad \dots(43)$$

This is the required Hamiltonian of the particle moving in MR fluid under the applied electromagnetic force.
From equation (36),

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \,, \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \dots(44)$$

The first equation of motion gives the same set of equations of motion described in equation (40).
Now consider second equation of motion,

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \dots(45)$$

Here $k = x, y, z$
Then,

$$\dot{p}_x = -\frac{\partial H}{\partial q_x} = \frac{q}{m}\frac{\partial}{\partial x}(p_xA_x + p_yA_y + p_zA_z) - \frac{q^2}{2m}\frac{\partial}{\partial x}(A_x^2 + A_y^2 + A_z^2) - q\frac{\partial \emptyset}{\partial x} \quad \dots(46)$$

$$\dot{p}_y = -\frac{\partial H}{\partial q_y} = \frac{q}{m}\frac{\partial}{\partial y}(p_xA_x + p_yA_y + p_zA_z) - \frac{q^2}{2m}\frac{\partial}{\partial y}(A_x^2 + A_y^2 + A_z^2) - q\frac{\partial \emptyset}{\partial y} \quad \dots(47)$$

$$\dot{p}_z = -\frac{\partial H}{\partial q_z} = \frac{q}{m}\frac{\partial}{\partial z}(p_xA_x + p_yA_y + p_zA_z) - \frac{q^2}{2m}\frac{\partial}{\partial z}(A_x^2 + A_y^2 + A_z^2) - q\frac{\partial \emptyset}{\partial z} \quad \dots(48)$$

The equation (46), (47), (48) can be written in vector form as,

$$\dot{p}_x = \frac{q}{m}\frac{\partial}{\partial x}(\bar{v}\,.\,\bar{A}) - q\frac{\partial \emptyset}{\partial x} \quad \dots(49)$$

$$\dot{p}_y = \frac{q}{m}\frac{\partial}{\partial y}(\bar{v}\,.\,\bar{A}) - q\frac{\partial \emptyset}{\partial y} \quad \dots(50)$$

$$\dot{p}_z = \frac{q}{m}\frac{\partial}{\partial z}(\bar{v}\,.\,\bar{A}) - q\frac{\partial \emptyset}{\partial z} \quad \dots(51)$$

Then above set of equations (49), (50), (51) can be expressed in vector form as,

$$\dot{\bar{p}} = -q\nabla\emptyset + q\nabla(\bar{v}\,.\,\bar{A}) \quad \dots(52)$$

Using the above equation we can determine the generalized momenta for single point particle of the MR fluid. In this way, we find the above set of equations for all particles of MR fluid and to explain the nature of the MR fluid under the action of electromagnetic forces.

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V. CONCLUSIONS

From a mechanics point of view, magnetorheological fluid can be viewed as a constrained motion of magnetic particles in MR fluid under the action of magnetic forces. The analytical mechanics is the most important area of research on MR fluids. In this paper, a brief introduction about Hamiltonian equations for magnetorheological fluid is presented. Then these concepts are utilized to derive Lagrange's equations from Hamiltonian equations for a system of magnetic particles. Here Hamiltonian formulation is developed for a single particle and to be extended for two or more system of the particles. This paper explains the mechanics of magnetic particles which controlled by an external magnetic force.

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